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Association Schemes and Graph Spectrum

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DEDICATION

This thesis is dedicated to my mother, who taught me that even the largest task can be accomplished if it is done one step at a time. It is also dedicated to my father, who taught me that the best kind of knowledge to have is that which is learned for its own sake.

Finally, this thesis is dedicated to all those who believe in the richness of learning

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Ala'a Majed AL-Mrayat

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LIST OF SYMBOLS

Symbol	Meaning
$V(\Gamma)$	Vertex set of Γ
$E(\Gamma)$	Edge set of Γ
p_n	Path of order n
$g(\Gamma)$	Shortest cycle in Γ
a_0	Vertex cover set of Γ
a_1	Edge cover set of Γ
b_0	Independent set of vertices
b_1	Independent set of edges
$A(\Gamma)$	Adjacency matrix
$B(\Gamma)$	Incident matrix
$\Gamma_1 \cup \Gamma_2$	Graph constructed from Vertex- disjoint Γ_1 and Γ_2
$\Gamma_1 + \Gamma_2$	Join of Γ_1 and Γ_2
K_n	Complete graph of order n
C_n	Cycle of order n
$K_{n,m}$	Complete bipartite graph
$S(1,n)$	Star graph of order n
$W(1,n)$	Wheel graph of order n
$\overline{\Gamma}$	Complement of Γ
l	Eigenvalue
$P_\Gamma(l)$	Characteristic polynomial Of graph Γ
$Spec(\Gamma)$	Spectrum of Γ
$Val(u_i)$	Degree of vertex u_i
$d(u,n)$	Distance between u and v
$\Omega \times \Omega$	Cartesian product of Ω
p_{ij}^k	Intersection numbers of The association scheme
$diam(\Gamma)$	Diameter of Γ
$d(\Gamma)$	Minimum valancy of Γ
D	Diameter of graph

LIST OF FIGURES

ABSTRACT

Association Schemes and Graph Spectrum

Ala'a Majed AL-mrayat

M'utah University, 2012

In this thesis, we will discuss the basic concepts of graphs, the representation of graphs by different types of matrices, the representation of graphs by characteristic polynomial and the spectrum of graphs. New idea of algebraic graph theory will be introduced such as association scheme and its relation with general geometrical properties of graphs.New results in this field illustrated by examples will be given.

ملخص

مخططات الارتباط و أطياف الأشكال

الآء ماجد المرايات

جامعة مؤتة، 2012

سنقوم في هذه الرسالة بدراسة المفاهيم الأساسية للبيانات (الأشكال) وتمثيل البيانات (الأشكال) بأنواع مختلفة من التراكيب والنماذج الرياضية كالمصفوفات ومميز كثير الحدود وأطياف الأشكال. أفكار جديدة لدراسة الارتباط بين الشكل الهندسي و التمثيل الهندسي للشكل ستطرح بالخصوص مخطط الارتباط وعلاقته بالأشكال. نتائج جديدة و أمثله موضحة ستقدم في هذا الحقل.

Chapter One
Introduction and Basic Definitions

1.1 An Overview

One of the most brilliant branches in mathematical area is called Graph Theory, which only came into existence during the first half of the 18^{th} century. The paper written by Leonhard Euler on the Seven Bridges of Konigsberg which is regarded as the first paper in the history of graph theory as well as the one written by Vander monde on the Knight problem curried on with the analysis situs initated by Leibniz. Euler's formula relating the number of edges, vertices, and faces of a convex polyhedron was studied and generalized by Cauchy and L'Huillier. This area didn't start to develop into an organized branch of mathematics until the second half of the 19^{th} century. Cayley was led by the study of particular analytical forms arising from differential calculus to study a particular class of graphs,trees. This study had many implications in theoretical chemistry there wasn't even a book on graph theory until the first half of the 20^{th} century. Since the second half of the 20^{th} century, however, the subject has exploded.

In particular, the term "Graph" was introduced by Sylvester in a paper published in 1878 in Nature, where he draws an analogy between "quantic invariants" and "co-variants" of algebra and molecular diagrams. Graphs are among the most ubiquitous models of both natural and human-made structures. They can be used to model many types of relations and process dynamics in physical, biological and social systems. Many problems of practical interest can be represented by graph. Graph Theory has a lot of applications. such as the computer science, mathematics, in sociology and in the analysis of dynamical properties of net works. Also graph theory is used to study molecules in chemistry and physics.

This thesis is mainly concerned with the investigation from the Association Schemes and graph Spectrum of certain classes of graphs. Also the relationship between graph and the characteristic polynomial of graph, by using matrix theory as a tool in graph theory and its application, we will introduce some new results concerning the graph structure and linear algebra.

Chapter 1, includes basic definitions and concepts of graph theory. It also defines some of the notation that will be used.

Chapter 2, we collect those results which are used regularly throughout or considered as general facts in related argument.

Chapter 3, we study the relations between the spectrum and graphical properties. Moreover the relationship between graph and characteristic polynomial of graph. New results have been achieved.

Chapter 4, we sets out to investigate (symmetric) association schemes Also, we well consider some relations between and among intersection numbers of association schemes and eigenmatrices which give a glance in algebraic graph theory.

1.2 Basic Definitions and Concepts

In this section, we present some basic definitions and concepts of Graph Theory which are needed later in this thesis.

Note : all definitions below are from (Harary,1972)

Definition 1.2.1A graph Γ =(V,E) consist of a finite non empty set of vertices V = V (Γ) to gather with a prescribed set of edges E = E (Γ).

Definition 1. 2.2 The number of vertices in Γ is called the order of Γ denoted by $|V(\Gamma)|$.

Definition 1.2.3 The number of edge in Γ is called size of Γ and denoted by $|E(\Gamma)|$.

Definition 1.2.4 The valency of vertex denoted by val $(v_i) \equiv$ number of edge incident to v_i .

Definition 1.2.5 A graph (Γ) is called k-regular if the valency of (v)=k for all vertices in Γ.

Definition 1.2.6 Two vertices (u,v) are said to be adjacent if they are joined by an edge e = (u,v). In addition , we said that v and e are incident also as e and u.

Definition 1.2.7 Two edges or more are incidents if they have a vertex in common with if not , they did to be independent edges.

Definition 1.2.8 A graph $\Gamma$is simple if it is undirected, has no loops, and no parallel edges.

Definition 1.2.9 A walke of a graph is an alternating sequence of vertices $u_0, x_1, ..., u_{n-1}, x_n, u_n$ beginning and ending with points in which each line is incident with two vertices immediately preceding and following it.

Definition 1.2.10 A trial is a walk with distinct edges.

Definition 1.2.11 A path is a walk with distinct vertices and edges.

Definition 1.2.12 A Path Graph, denoted by P_n is a graph consisting of a path with following properties:

(i) the number of vertices is n.

(ii) the number of edges is (n-1).

(iii) the degree of a start vertex and end vertex is 1.

(iv) degree of all vertices (except start vertex and end vertex) is 2.

Definition 1.2.13 The length of the shortest cycle in graph Γcalled the girth of graph and denoted by $g(\Gamma)$.

Definition 1.2.14 A vertex and edge are said to be cover each other if they are incident.

Definition 1.2.15 A vertex cover set for a graph Γ is a set of vertices which cover all the edges of a graph Γ and the smallest number of vertices that cover the graph denoted by (a_0),while the edge cover set of Γ is a set of edges that cover all vertices of Γ and the smallest number of edges that cover the vertices denoted by (a_1).

Definition 1.2.16 The set of all vertices in Γ in which every two vertices are not adjacent is called independent set of vertices , and the maximum number of independent vertices denoted by $B_0(\Gamma)$, while the maximum number of independent edges denoted by $B_1(\Gamma)$.

Definition 1.2.17 Let Γ be a simple graph, the line graph of Γ denoted by $L(\Gamma)$ which is a graph with $V(L(\Gamma)) = E(\Gamma)$ and two vertices are adjacent if they are incident in Γ.

Definition 1.2.18 A graph Γ is connected if there exists at least one path between any two vertices .

Definition 1.2.19 The adjacency matrix $A(\Gamma) = [a_{ij}]_{n\times n}$ of a labeled graph Γ with n vertices $\{v_1, v_2, v_3, ..., v_n\}$ is an $n\times n$ matrix in which

$$a_{ij} = \begin{cases} 1 \quad , \quad if \quad u_i \; adjacent \quad with\, u_j \\ 0 \quad , \quad otherwise \end{cases}.$$

Thus , there is one to one corresponding between labeled (simple) graphs with n vertices and $n\times n$ symmetric binary matrices with zero diagonal.

Definition 1.2.20 The determinant of matrix $|l\mathrm{I} - A|$ is called the characteristic polynomial of a graph Γ where A is the adjacency matrix

of a graph and denoted by $P_\Gamma(l) = a_n l^n + a_{n-1} l^{n-1} + a_{n-2} l^{n-2} + ... + a_0$.

Definition 1.2.21 The eigenvalues of the adjacency matrix A are the zero of $|l\mathrm{I} - A|$, and the spectrum of the graph Γ consist of eigenvalues of the adjacency matrix A together with their Multiplicities, and it denoted by spec (Γ).

Definition 1.2.22 Let Γ be a graph with $V(\Gamma)$ and $E(\Gamma)$, then the incident matrix of Γ denoted by $B[\Gamma] = B = [b_{ij}]_{|V(\Gamma)||E(\Gamma)|}$

$$b_{ij} = \begin{cases} 1, \quad if \quad u_i \in e_j \\ 0, \quad otherwise \end{cases}.$$

Definition 1.2.23 Two graphs Γ_1 and Γ_2 are isomorphic if there exist a one to one corresponding between there vertices which preserve the adjacency and denoted by $\Gamma_1 \cong \Gamma_2$

Definition 1.2.24 A component of Γ is a maximal connected sub graph of Γ, in the other words ,a connected subgraph H is a component of a graph Γ is not a proper sub graph of any connected sub graph of Γ.

Definition 1.2.25 A graph Γ is complete graph if $|v(k_n)| = n$ and $E(k_n) = \frac{1}{2}(n^2 - n)$, and every vertex is adjacent to all other vertices.

Definition 1.2.26 A complete bigraph $K_{m,n}$ is a graph with two disjoint set of vertices A, B such that every vertex in A adjacent to all vertices in B,and the sets are disjoint.

Definition 1.2.27 A graph consists of n isolated vertices called a scuter graph.

Definition 1.2.28 A star graph, is a graph consisting of n+1 vertices and n edges and denoted by $S(1,n)$ or by S_n with n vertices with following properties :

(i) It has a root vertex adjacent with all vertices.
(ii) The outer vertices are not adjacent.
(iii) The valency of root = n.

Definition 1.2.29 A wheel graph is a graph consisting of n vertices denoted by W_n with n vertices or by $W(1,n)$ with n+1 Vertices with following properties:

(i) It has a root vertex adjacent with all vertices with valency equal n
(ii)The number of edges is 2n
(iii)The outer vertices has valency is 3

Definition 1.2.30 [Union $(\Gamma_1 \cup \Gamma_2)$] Let Γ_1 and Γ_2 be two graphs. Then $\Gamma_1 \cup \Gamma_2$ is a graph with $V(\Gamma_1 \cup \Gamma_2) = V(\Gamma_1) \cup V(\Gamma_2)$ and $E(\Gamma_1 \cup \Gamma_2) = E(\Gamma_1) \cup E(\Gamma_2)$.

Definition 1.2.31 [Join $(\Gamma_1 + \Gamma_2)$] Let Γ_1 and Γ_2 be two graphs. Then $(\Gamma_1 + \Gamma_2)$ is a graph with $V(\Gamma_1 + \Gamma_2) = V(\Gamma_1) \cup V(\Gamma_2)$ and $E(\Gamma_1 + \Gamma_2)$ consist of the edges set union with all edges from Γ_1 into Γ_2.

Definition 1.2.32 The valency of R_i denoted by n_i is the number of $z \in \Omega$ with $(x, z) \in R_i$.

Chapter Two
The earlier studies

In this chapter we collect most of the results which are considered as known facts and used throughout this thesis.

Theorem 2.1 [Fielder.1973] For any graph Γ , $VC(\Gamma) \le EC(\Gamma) \le d(\Gamma)$, where $d(\Gamma)$ is the minimum valency of Γ .

Corollary 2.1 [Fielder.1973] Let Γ be a K regular graph and let $Spec(\Gamma) = \{l_1, l_2, ..., l_n\}$,such that $l_1 \ge l_2 \ge ... \ge l_n$. Then $VC(\Gamma) \le l_1$.

Theorem 2.2 [Fielder1973] Let $l_1 = r$, $l_1, l_2, ..., l_n$ be the spectrum of a graph Γ, r being the index of Γ then Γ is regular if and only if $\frac{1}{n}\sum_{i=1}^{n} l_i^2 = r$.

Theorem 2.3[Cvetkovic.1971] Let Γ be a connected regular graph of valency k on n vertices with transitive automorphism group, and let l be a simple eigenvalue of Γ.Then

$$l = \begin{cases} k & , \ if \quad n \quad is \quad odd \\ 2q - k & , \ if \quad n \quad is \quad even \end{cases}$$

where, $q \in \{0, 1, \mathbf{K}, k\}$.

Theorem 2.4[G.Chartrand and P.Zhang.2005] (The First Theorem of Graph Theory)

if Γ is a graph of size m, then $\sum_{v \in V(\Gamma)} \deg v = 2m$.

Theorem 2.5[G.Chartrand and P.Zhang2005] If Γ is a disconnected graph, then $\bar{\Gamma}$ is connected.

Theorem 2.6[E.Nering1970] (Hamilton –Cayley theorem)

If A is a square matrix and $f(l)$ is its characteristic polynomial .Then $f(A) = 0$.

Theorem 2.7 [Cvetkovic, Doob and Sach1978] If a connected graph Γ has exactly m distinct eigenvalues then its diameter $diam(\Gamma)$ satisfies the inequality $diam(\Gamma) \le m - 1$.

Corollary 2.2 [N.L.Biggs1974] A connected graph Γ with n vertices and diameter d has at least $d + 1$, and at most n distinct eigenvalues .

Chapter Three
Graph Spectrum

In this chapter we will discuss the graph spectrum and the relations between the spectrum of graphs , graphical properties and the related characteristic polynomial of graphs .

3.1 Matrices Associated to Graphs

As we know a graph Γ can be defined by two sets namely its vertex $V(\Gamma)$ and edge set $E(\Gamma)$ or by a diagram. A graph can also be described by a matrix. Let Γ be a graph of order n and size m,where $V(\Gamma)=\{v_1,v_2,...,v_n\}$ and $E(\Gamma)=\{e_1,e_2,...,e_m\}$ the adjacency matrix of Γ is the $n\times n$ matrix $A=[a_{ij}]$ where

$$a_{ij}=\begin{cases}1 \quad , \quad if \quad u_i,u_j \quad \in E(\Gamma) \\ 0 \quad , \quad otherwise\end{cases} \quad .$$

$n\times m$ is the Γ while the incidence matrix of

$$b_{ij}=\begin{cases}1 \quad , \quad if\ u_i \quad is \quad incident \quad with \quad e_j \\ 0 \quad , \quad otherwise\end{cases} \quad . \text{ ,where } B=[b_{ij}] \text{matrix}$$

Example 3.1.1: Let $\Gamma=K_4$ be a graph. Then the adjacency matrix is

$$A=\begin{bmatrix}0&1&1&1\\1&0&1&1\\1&1&0&1\\1&1&1&0\end{bmatrix}$$

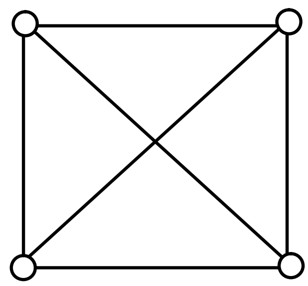


Figure 3.1: K_4

Note 3.1.1: These matrices are dependent on how the vertices and edges of Γ are labeled. In any case, the adjacency matrix is a symmetric $n\times n$ matrix where every entry on the main diagonal is 0.

Example 3.1.2: The two matrices

$$B=\begin{bmatrix}1&1&1&0&0&0\\1&0&0&1&0&0\\0&1&0&0&1&0\\0&0&1&1&1&1\\0&0&0&0&0&1\end{bmatrix} \quad , A=\begin{bmatrix}0&1&1&1&0\\1&0&0&1&0\\1&0&0&1&0\\1&1&1&0&1\\0&0&0&1&0\end{bmatrix}$$

are the adjacency matrix and incidence matrix of a graph .

Example 3.1.3: Let Γ be the path P_3 with three vertices and two edges thus the adjacency matrix is

$$A=\begin{bmatrix}0&1&0\\1&0&1\\0&1&0\end{bmatrix}$$

Theorem 3.1.1[G.Chartrand and P.Zhang2005] Let Γ be a graph with vertex set $V(\Gamma)=\{v_1,v_2,...,v_n\}$ and adjacency matrix $A=[a_{ij}]$.Then the entry a_{ij}^K in row i and column j of A^K is the number of $v_i - v_j$ walks of length k in Γ.

Proof: We proceed by induction .Let $A=A^1$ each entry $a_{ij}=a_{ij}^{(1)}$ of A is either 1 or 0 according to whether v_iv_j is not an edge of Γ. Hence $a_{ij}^{(1)}$ gives the number of $v_i - v_j$ walks of length 1 in Γ. Assume, for appositive integer K, that the number of $v_i - v_j$ walks of length k in Γ is given by $a_{ij}^{(k)}$ from the definition of matrix multiplication, the $(i,j)-$ entry $a_{ij}^{(k+1)}$ in A^{k+1} is the dot product of row i of A^k and column j of A, that is

$$a_{ij}^{(k+1)}=\sum_{t=1}^{n}a_{it}^{(k)}a_{tj}=a_{i1}^{(k)}a_{1j}+a_{i2}^{(k)}a_{2j}+...+a_{in}^{(k)}a_{nj}\text{..........................}(3.1)$$

every $v_i - v_j$ walk W of length $k+1$ is produced by beginning with a $v_i - v_t$ walk W' of length K for some vertex v_t adjacent to v_i

and then follow W' by v_j. By the induction hypothesis, the

number of $v_i - v_t$ walks of length K is $a_{it}^{(k)}$, while v_t is adjacent to

v_j if and only if $a_{tj}=1$.Hence by (3.1) , $a_{ij}^{(k+1)}$ does indeed provide

the number of $v_i - v_j$ walks of length $k+1$ in Γ.

Note 3.1.2: In view of Theorem 3.1.1, $a_{ii}^{(2)}=\deg v_i$ and $a_{ii}^{(3)}$ is twice the number of triangles in Γ that contain v_i. Indeed, knowing what each entry of A^K represents allows us to compute powers of adjacency matrices , at least small power of adjacency matrices of graphs of small orders without actually multiplying matrices.

Example 3.1.4: For the graph Γ of Example 3.1.2, the entry $a_{41}^{(3)}$ of A^3 is the number of distinct $v_4 - v_1$ walks in Γ of length 3. Since there are six $v_4 - v_1$ walks of length 3 in Γ namely:

v_4, v_5, v_4, v_1 , v_4, v_2, v_4, v_1 , v_4, v_3, v_4, v_1 , v_4, v_1, v_2, v_1

, v_4, v_1, v_3, v_1 v_4, v_1, v_4, v_1

It follows that $a_{41}^{(3)} = 6$ as we can see

$$A^2 = \begin{bmatrix} 3 & 1 & 1 & 2 & 1 \\ 1 & 2 & 2 & 1 & 1 \\ 1 & 2 & 2 & 1 & 1 \\ 2 & 1 & 1 & 4 & 0 \\ 1 & 1 & 1 & 0 & 1 \end{bmatrix} \text{ and } A^3 = \begin{bmatrix} 4 & 5 & 5 & 6 & 2 \\ 5 & 2 & 2 & 6 & 1 \\ 5 & 2 & 2 & 6 & 1 \\ 6 & 6 & 6 & 4 & 4 \\ 2 & 1 & 1 & 4 & 0 \end{bmatrix}$$

(powers of an adjacency matrix)

3.2 Characteristic Polynomial of well known Graph

The characteristic polynomial is the polynomial of the characteristic equation $\det(A - l\mathrm{I}) = 0$, where A is a square matrix and I is the identity matrix The characteristic polynomial of a 2×2 matrix

$$P_2(l) = l^2 - l(a_{11} + a_{22}) + (a_{11}a_{22} - a_{12}a_{21})$$

which can be written in the particularly nice form

$$P_2(l) = l^2 - lTr(A) + \det(A)$$

where Tr(A) is the matrix trace of A and det(A) is its determinant.

Similarly, the characteristic polynomial of connected graph with three vertices is

$$P_3(l) = -l^3 + Tr(A)\,l^2 + \frac{1}{2}(a_{ij}a_{ji} - a_{ii}a_{jj})\,l + \det(A)$$

which can also be written explicitly in terms of traces as

$$P_3(l) = l^3 + Tr(A)\,l^2 + \left[Tr^2(A) - Tr(A^2)\right] l + \left[Tr^3(A) + 2Tr(A^3) - 3Tr(A)\right]$$

In general , the characteristic polynomial has the form

$$P(l) = \det(l\mathrm{I} - A) = l^n - a_1 l^{n-1} + \ldots + (-1)^n a_n$$

where $a_1 = \sum a_{ii}$ is the matrix trace Tr(A) of the matrix A$a_n = \det(A)$ and a_i is the sum of the i—rowed diagonal, minors of the matrix A .The characteristic polynomial of a graph Γ is defined as the characteristic polynomial of its adjacency matrix.

Theorem 3.2.1[N.L.Biggs1974] The characteristic polynomial of graph Γ is

$$P_\Gamma(l) = a_n l^n + a_{n-1} l^{n-1} + a_{n-2} l^{n-2} + \ldots + a_0 .$$

Then we have

(i) $a_{n-1} = 0$

Γ is the number of edge in $-a_{n-2} =$ (ii)

(iii) $-a_{n-3} =$ is twice number of triangle in Γ

Example 3.2.1: Let Γ be $K_{2,2}$ graph. Then the adjacency matrix is

$$A = \begin{bmatrix} 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \end{bmatrix}$$

The characteristic polynomial is $P_\Gamma(l) = -4l^2 + l^4$. And the number of edges is $a_2 = 4$ also the number of triangles is equal 0 since $a_3 = 0$.

Proposition 3.2.1[N.L.Biggs1974]

The largest eigenvalue implies that the eigenvalue of Γ are $l_0 \geq \ldots \geq l_{n-1}$. where Γ has $n \times m$ we obtain $\sum l_i = 0$ and $\sum l_i^2 = 2m$

it follows that $l_0 \leq \left(\sqrt{\frac{2m(n-1)}{n}} \right)$.

proposition 3.2.2 [W.Cheney and D.Kincaids 2009]

all the roots of the characteristic polynomial of real symmetric matrix are real numbers.

graph. Then the adjacency matrix is K_2 be Γ **Example 3.2.2:** Let

$A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$. And the characteristic polynomial is $P(l) = l^2 - 1$

Note 3.2.1: characteristic polynomials are not diagnostic for graph isomorphism,(that is two non isomorphic graphs may share the same characteristic polynomial).

Example 3.2.2:

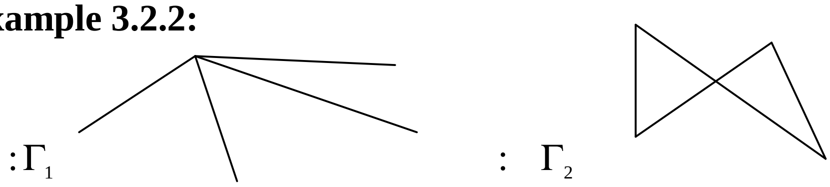


Figure 3.2: Two graphs Γ_1 and Γ_2 are isomorphic

Which are two graph on five nodes both of which have characteristic polynomial $4l^3 - l^5$.

Theorem 3.2.2 [N.L.Biggs1974] If Γ is k -regular graph. Then

(i) k is an eigenvalue of Γ

(ii) If Γ is connected, then $m[K]=1$

(iii) For any other eigenvalue l of Γ, l less than or equal k .

3.3 Spectrum of well known Graphs

The spectrum of a graph Γ is by definition the spectrum of the Adjacency matrix A, that is its set of eigenvalues together with their multiplicities.

Example 3.3.1: Let Γ be the triangle with adjacency matrix

$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$.Then A has characteristic polynomial

$P_\Gamma(l) = l^3 - 1$, and the spectrum $1, w, w^2$

Proposition 3.3.1[N.L.Biggs1974] If the spectrum of Γ is

$$Spec(\Gamma) = \begin{bmatrix} K & l_1 & . & . & . & l_{i-1} \\ 1 & m_1 & . & . & . & m_{i-1} \end{bmatrix}$$

Then $SpecL(\Gamma) = \begin{bmatrix} 2K-2 & K-2+l_1 & . & . & . & K-2+l_{i-1} & -2 \\ 1 & m_1 & & . & . & . & m_{i-1} & m-n \end{bmatrix}$

Note 3.3.1: A graph Γ is called regular of degree or (valency) k , when every vertex has precisely k neighbors. So Γ is regular of degree k precisely when its adjacency matrix A has row sums k ,(i.e when $AI = KI$)

If Γ is regular of degree K, then for every eigenvalue l we have $|l| \le K$.

Example 3.3.2: Let Γ be K_1 graph. Then $Spec(K_1) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

is a trivial graph and the characteristic polynomial is $-l$

Theorem 3.3.1[N.L.Biggs1974] The spectrum of the complete graph is $P(K_n, l) = (lI - A)$

$$= \det \begin{bmatrix} l & -1 & -1 & \cdots & -1 \\ 0 & \frac{(l+1)(l-1)}{l} & \frac{-(l+1)}{l} & \cdots & \frac{-(l+1)}{l} \\ 0 & 0 & \frac{(l+1)(l-2)}{(l-1)} & \cdots & \frac{-(l+1)}{l} \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & 0 & \cdots & \frac{(l+1)(l-(n-1))}{(l-(n-2))} \end{bmatrix}$$

$$p(K_n, l) = (l+1)^{n-1}(l-n+1)$$

$$Spec(K_n) = \begin{bmatrix} n-1 & -1 \\ 1 & n-1 \end{bmatrix}$$

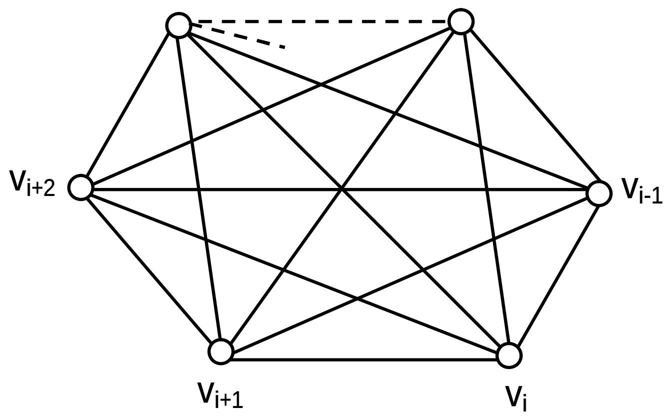


Figure3.3: K_n

Example 3.3.3: The spectrum of K_3

by the above theorem is $Spec(K_3) = \begin{bmatrix} 2 & -1 \\ 1 & 2 \end{bmatrix}$

proposition 3.3.2 [N.L.Biggs 1974] Spectrum of cycle graph

$$Spec(C_n)=\begin{cases} 2 & 2cos(n-1)\frac{p}{n} & . & . & . & 2cos(n-1)\frac{p}{n} & n & is\ odd \\ 1 & 2 & . & . & . & 2 \end{cases}.$$

$$Spec(C_n)=\begin{cases} 2 & 2cos\,2\frac{p}{n} & . & . & . & 2cos(n-2)\frac{p}{n} & n & is\ even \\ 1 & 2 & . & . & . & 2 \end{cases}.$$

Example 3.3.4: Let Γ be C_6 graph. Then the adjacency matrix is

$$A=\begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

The characteristic polynomial is $P(l)=l^6-6l^4+9l^2-4$

The characteristic form is $(l+1)^2(l-1)^2(l+2)(l-2)$

Thus the eigenvalues are $x=z$

The spectrum is $Spec(C_6)=\begin{bmatrix} -2 & -1 & 1 & 2 \\ 1 & 2 & 2 & 1 \end{bmatrix}$

Theorem 3.3.2 : Let C_n be a cycle graph. Then the line graph of C_n is $L(C_n)=C_n$

Proof : Let C_n be a cycle graph , C_n is connected path graph with n – vertices and $|E(C_n)|=n$ since its connected then the end vertex is the same (by the definition of line graph) $L(C_n)=C_n$.

Remark 3.3.1: The spectrum of $L(C_n)$ is the same as the original graph .

Theorem 3.3.3 : Let $S(1,n)$ be a star graph. Then the line graph of $L(S(1,n))$ is K_n

Proof : Let $S(1,n)$ be a star graph ,with $|E(S(1,n))|=n$ (since all edges has vertex in common (root)) . Thus the vertices in $L(S(1,n))$ adjacent with all rest vertices thus $L(S(1,n))=K_n$.

Theorem 3.3.4 : Let $S(1,n)$ be a star graph , then the spectrum of $S(1,n)$ is

$$Spec(S(1,n)) = \begin{bmatrix} \sqrt{n-1} & 0 & -\sqrt{n-1} \\ 1 & n-2 & 1 \end{bmatrix}$$

Proof : Let A be the adjacency matrix of the star graph $S(1,n)$, thus A is an $n \times n$ matrix with the zero entries, in the diagonal

$$A = \begin{bmatrix} 0 & 1 & 1 & \dots & \dots & 1 & 1 \\ 1 & 0 & 0 & \dots & \dots & 0 & 0 \\ . & . & . & . & . & . & . \\ . & 0 & . & . & . & . & 0 \\ 1 & 0 & 0 & . & . & . & 0 \end{bmatrix}$$

Then the characteristic polynomial of $S(1,n)$ is $p_{S(1,n)}(l) = l^{n-2}(l^2 - (n-1))$

Thus the root $l = 0$ is repeated $n-2$ times and $\sqrt{n-1}$, $-\sqrt{n-1}$ are repeated once .

Corollary 3.3.1: Let $L(S(1,n))$ be a line graph of $S(1,n)$. Then the spectrum of $L(S(1,n))$ is $Spec(L(S(1,n))) = \begin{bmatrix} n-1 & -1 \\ 1 & n-1 \end{bmatrix}$

Proof : The result holds by Theorem3.3.1 and Theorem3.3.3.

Example 3.3.5 : Let $S(1,4)$ be a star graph .Then by applying theorem 3.3.4 we have

$$Spec(S(1,4)) = \begin{bmatrix} 2 & 0 & -2 \\ 1 & 3 & 1 \end{bmatrix}$$

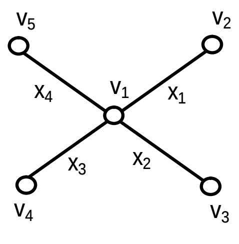


Figure 3.4:star

proposition 3.3.3[N.L.Biggs1974] The spectrum of $K_{m,n}$ is

$$Spec(K_{m,n}) = \begin{pmatrix} \sqrt{mn} & 0 & -\sqrt{mn} \\ 1 & m+n-2 & 1 \end{pmatrix}$$

Corollary 3.3.2 : The spectrum of $K_{n,n}$ is

$$Spec(K_{n,n}) = \begin{bmatrix} n & 0 & -n \\ 1 & 2n-2 & 1 \end{bmatrix}$$

Example 3.3.6 : The spectrum of $K_{3,3}$ is

$$Spec(K_{3,3}) = \begin{bmatrix} 3 & 0 & -3 \\ 1 & 4 & 1 \end{bmatrix}$$

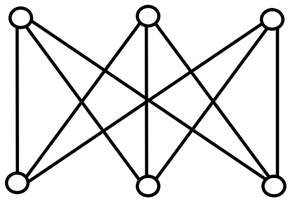


Figure 3.5 $K_{3,3}$

Example 3.3.7: The spectrum of $K_{2,2}$ is

$$Spec\left(K_{2,2}\right) = \begin{pmatrix} -2 & 0 & 2 \\ 1 & 2 & 1 \end{pmatrix}$$

Example 3.3.8: The spectrum of $K_{8,2}$ is

$$Spec\left(K_{8,2}\right) = \begin{pmatrix} 4 & 0 & -4 \\ 1 & 8 & 1 \end{pmatrix}$$

Example 3.3.9: Let Γ be $S\,(1,4)$ .Then the adjacency matrix is

$$A = \begin{bmatrix} 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{bmatrix}$$

The characteristic polynomial $P\,(l) = 4l^3 - l^5$

The eigenvalue are $l = -2, 0, 0, 0, 2.$

The spectrum is $Spec\left(S\,(1,4)\right) = \begin{pmatrix} -2 & 0 & 2 \\ 1 & 3 & 1 \end{pmatrix}$

Example 3.3.10: Let $S(1,9)$ be a star graph. Then the spectrum is

$$Spec(S(1,9)) = \begin{pmatrix} 3 & 0 & -3 \\ 1 & 8 & 1 \end{pmatrix}$$

Theorem 3.3.5 : Let p_n be a path graph .Then the line graph of p_n is p_{n-1}.

Proof : Let p_n be a path graph with n-vertices, every vertex adjacent with the vertex which follow it . By the line graph definition we obtain that the number of edge in p_n is (n-1) ,so the number of vertices in $L(p_n) = p_{n-1}$,

every vertex in $L(p_n)$ adjacent with the vertex which follow it. Thus $L(p_n) = p_{n-1}$.

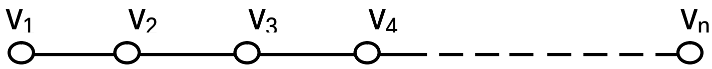


Figure 3.6:
p_n graph

Theorem 3.3.6: Let p_n be a path graph. Then the spectrum of p_n

$$Spec(P_n) = \begin{pmatrix} 0 & \sqrt{n-2} & -\sqrt{n-2} \\ n-2 & 1 & 1 \end{pmatrix}$$

Proof : Let A be the adjacency matrix of the graph p_n.Thus A is a $n \times n$ matrix with the zero entries ,in the diagonal

$$A = \begin{bmatrix} 0 & 1 & 0 & . & . & . & 0 & 0 \\ 1 & 0 & 1 & . & . & . & 0 & 0 \\ 0 & . & 0 & . & . & . & 0 & 0 \\ . & . & . & 0 & . & . & . & . \\ . & . & . & . & 0 & . & . & . \\ . & . & . & . & . & 0 & . & 0 \\ 0 & 0 & . & . & . & . & 0 & 1 \\ 0 & 0 & . & . & . & . & 1 & 0 \end{bmatrix}$$

Then the characteristic polynomial is

$$p_{p_n}(l) = |A - lI| = l^{n-2}\left(l^2 - (n-2)\right),$$

Consequently the root zero is repeated $n-2$ times and $\sqrt{n-2}$, $-\sqrt{n-2}$ are repeated once .

Example 3.3.11: The spectrum of p_{18} is

$$Spec(p_{18}) = \begin{bmatrix} 0 & 4 & -4 \\ 16 & 1 & 1 \end{bmatrix}$$

Remark 3.3.2 : For every two adjacent vertices u and n in a connected graph $|e(u)-e(n)| \le 1$

Proof : Assume that $e(u) \ge e(n)$

Let c be a vertex that is furthest from u so $d(u,c) = e(u)$

$e(u) = d(u,c) \le d(u,n) + d(n,c)$ By the triangle inequality

$\le 1 + e(n)$

Hence , $e(u) \le 1 + e(n)$, which implies that $0 \le e(u) - e(n) \le 1$

$|e(u)-e(n)| \le 1.$

Theorem 3.3.7 : All cycles in the $K_{n,m}$ are even

Proof : Let Γ be a $K_{n,m}$ graph so the set of vertices divided into two disjoint sets A_1 and A_2 no vertices in A_1 adjacent with each others and so with respect to A_2

without loss of generality, let $u_i, u_j \in A_1$ and $n_i, n_j \in A_2$ u_i adjacent n_i , and n_j adjacent to n_i , and n_j .
Then a cycle of length 4 constructed. Following the same argument for the rest vertices cycle of even length constructed and this contradicts the definition of bigraph .

Theorem 3.3.8 : Let $char(\Gamma) = l^n + a_{n-1} l^{n-1} + \ldots + \sum_{i=4}^{r} a_i$ where i even. Then

Γ is $K_{n,m}$ graph; where $r = n + m$.

Proof : Let $char(\Gamma) = l^n + a_{n-1} l^{n-1} + \ldots + \sum_{i=4}^{r} a_i$.

By theorem number of cycles of length i. Since, all odd $i \geq 3$ are vanished, and all indices factories a_i are even. So all cycle in Γ are even. Consequently Γ is a bigraph by Theorem 3.3.7.

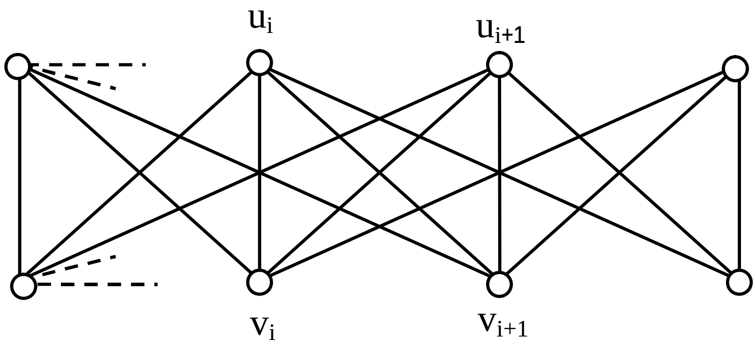


Figure 3.7: $K_{n,m}$

Chapter Four
Association Schemes

Association schemes were originally introduced by Bose and his co-workers in the design of statical experiments. Since that point of inception, the concept has proved useful in the study of group actions, in algebraic graph theory, in algebraic coding theory, and in areas far afield as Knot theory and numerical integration. This branch of the theory, viewed in this surveys as the "commutative case". The most important association schemes are the Hamming and Johnson schemes. Moreover in this chapter we sets out to investigate (symmetric) association schemes. Such schemes are essentially partitions of a complete graph into regular sub graphs which are interrelated in a specific way. also we well consider some relations between and among intersection numbers of association schemes and eigenmatrices. New and tangible results illustrated by examples will be introduced.

4.1. Construction of Association Schemes

Definition 4.1.1 : Let Ω be a finite set of points . An association scheme with s classes on a finite set Ω is a partition of $\Omega\times\Omega$ into sets $R_0, R_1, ..., R_s$ (called associate classes) such that .

$(R_0 = Diag\,(\Omega)$ i)

$(i = 1,...,s$ is symmetric for R_i ii)

such that for $P_{i,j}^k$ there is an integer $\{0,...,s\}$ in i, j, k (iii) For all

$\left|\{z \in \Omega : (x,z) \in R_i\, and\, (z,y) \in R_j\}\right| = P_{i,j}^k$, $(x,y) \in R_k$ any

Note 4.1.1

(i) Two points x and y are called i-th associates if $(x,y) \in R_i$ and so y and x since R_i is symmetric . Consequently, every pair of points are i-th associates for exactly one i. (ii)
Each point is its own zeroth associate. (iii)
Distinct points are not zeroth associates. (iv) If x and y k-th associates (i.e. $(x,y) \in R_k$), then the number of z which are both i-th associates of x and j-th associates of y is a constant P_{ij}^k .

Theorem 4.1.1: Let Ω be a finite set and $R_0, ..., R_s$ be a partition of $\Omega\times\Omega$. Then

(i) $P_{i,j}{}^0 = 0$ if $i \neq j$

Proof : By definition ,for all $(x,y) \in R_0$, $P_{i,j}^0$ is the number of z such that $(x,z) \in R_i$ and $(z,y) \in R_j$, but $R_0 = Diag\,(\Omega) = \{(x,x) : x \in \Omega\}$

So there is no such z satisfies the above conditions, thus $P_{i,j}{}^0 = 0$.

(ii) $P_{i,0}{}^{k}=0$ if $i\neq k$ and $P_{0,j}{}^{k}=0$ if $j\neq k$

Proof: For all $(x,y)\in R_k$,$P_{i,0}{}^{k}$ is number of z such that $(x,z)\in R_i$ and $(z,y)\in R_0$ .Since, $R_0=Diag(\Omega)$ i.e. $R_0=\{(y,y):y\in\Omega\}$,

so $(z,y)\notin R_0, P_{i,0}{}^{k}=0$, similarly $P_{0,j}{}^{k}=0$.

(iii) $P_{0,j}{}^{j}=1$ and $P_{i,0}{}^{i}=1$.

Proof : $P_{0,j}{}^{j}$ is equal to number of $z\in\Omega$ such that $\forall(x,y)\in R_j$, $(x,z)\in R_0$ and $(z,y)\in R_j$, since $(x,x)\in R_0, x=z$ so $(x,y)\in R_j$.

So, number of such z is one . Hence, $P_{0,j}{}^{j}=1$

Similarly, $P_{i,0}{}^{i}=1$.

Note 4. 1.2:

$P_{i,i}^{0}=n_i$,n_i called the valency (degree) of the i-th associates

$P_{i,i}^{0}=\left|\{z\in\Omega:(x,z)\in R_i,(z,y)\in R_i\forall(x,x)\in R_0\}\right|$

Theorem 4.1.2 : Let Ω be a set of points. Then every point of Ω has $P_{i,i}^{0}$ i-th associates, $P_{i,i}^{0}=n_i$, n_i equal the valency of the i-th associates.

Proof : Let Ω be a set of points and $R_0,\ldots,R_s$ be a partition of $\Omega\times\Omega$.

$\forall(x,x)\in R_0$ $P_{i,i}^{0}=\{z\in\Omega:(x,z)\in R_i,(z,y)\in R_i\}$,

Since , R_i is symmetric $\forall(x,z)\in R_i$, $(z,x)\in R_i$

$z=\frac{1}{2}|R_i|$.

Example 4.1.1: C_4 - graph

$\Omega=V(C_4)=\{v_1,v_2,v_3,v_4\}$

$R_0=\{(v_1,v_1),(v_2,v_2),(v_3,v_3),(v_4,v_4)\}$

edges (adjacent vertices) R_1

$R_1=\{(v_1,v_2),(v_1,v_4),(v_2,v_3),(v_3,v_4),(v_2,v_1),(v_4,v_1),(v_3,v_2),(v_4,v_3)\}$

R_2 with no edges

$R_2=\{(v_1,v_3),(v_2,v_4),(v_3,v_1),(v_4,v_2)\}$

So, $\{R_0,R_1,R_2\}$ are partition of $\Omega\times\Omega$

$\Omega=V(C_4)$

,$P_{00}^{0}=1$, $P_{1,1}^{0}=2$, $P_{2,2}^{0}=1$.Since, for any $(x,x)\in R_0$.

$P_{i,i}^{0}$ equal number of $z\in\Omega$ such that $(x,z)\in R_i$,$P_{2,1}^{0}=0$,$P_{1,2}^{0}=0$

For any $(x,x)\in R_0$, $(x,z)\in R_1,(z,x)\in R_2$. Does not exist since if $(x,z)\in R_0,(z,x)\in R_0$ and $(z,x)\notin R_1$

$P_{0,1}^{1}=1$, $P_{0,2}^{2}=1$, $P_{1,1}^{1}=0$, $P_{2,2}^{2}=0$.

For any $(x,y)\in R_i$, $(x,z)\in R_1$ and $(z,x)=(x,z)\in R_1, P_{1,2}{}^{1}=3$

Example 4.1.2: Let Γ be the complete graph K_4, $\Omega=V(K_4)$ and let R_1

consists of edges of K_4 (that is R_1 consists of pairs of adjacent vertices in K_4).

$$R_1 = \begin{Bmatrix} (v_1,v_2),(v_1,v_4),(v_2,v_3),(v_3,v_4) \\ (v_2,v_1),(v_4,v_1),(v_3,v_2),(v_4,v_2) \\ (v_3,v_1),(v_1,v_3),(v_4,v_3),(v_2,v_4) \end{Bmatrix}$$

R_2 consists of non adjacent vertices no edges. Then $R_2 = f$

Example 4.1.3: Let Γ be a Petersen graph (figure 4.1) $\Omega = V(\Gamma)$, and Δ be a partition of $\Omega\times\Omega$, where $\Delta = \{R_0, R_1, R_2\}$,

$R_0 = \{(v_i, v_i) : \forall v_i \in \Omega\}$.

$R_1 =$ The edges of Γ = the set of adjacent pair of vertices.

$R_2 =$ The non-edges of Γ =the set of non adjacent vertices of Γ

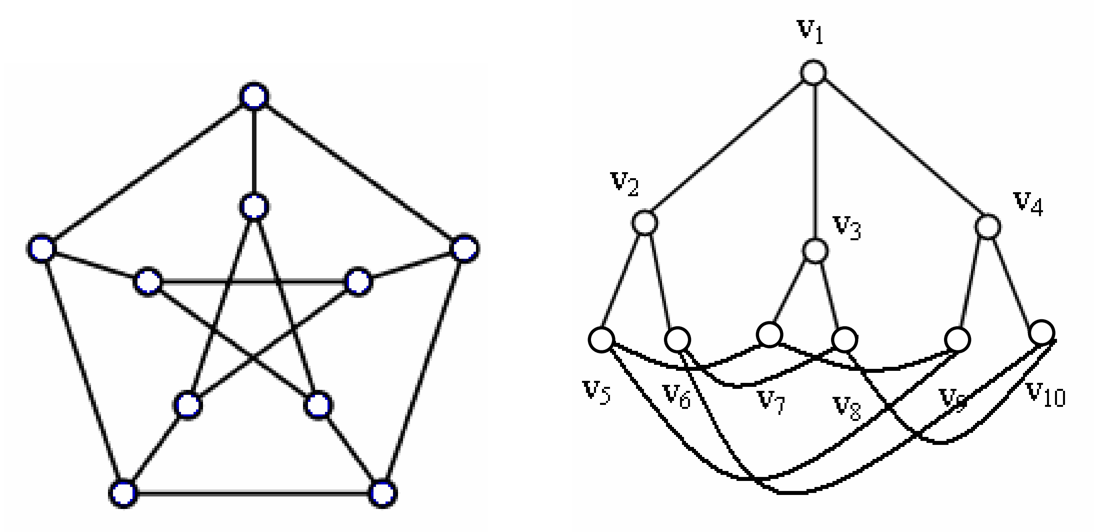


Figure 4 .1 : the Petersen graph

Inspection of the graph shows that Γ is a 3-regular graph with $|V(\Gamma)| = 10$ and $|E(\Gamma)| = 15$ and $g(\Gamma) = 5$, Γ has no C_3 subgraphs

Theorem4.1.3: Let Ω be a finite set with s classes. Then $\sum_{i=0}^{s} P_{i,i}{}^{0} = |\Omega|$

Proof : Let $\Delta = \{R_0, R_1, ..., R_s\}$, let $(x,x) \in R_0$, but $P_{i,i}^0$ equal number of $z \in \Omega$ such that $(x,z) \in R_i$ and since, $P_{i,i}^0 = n_i$ (by definition of n_i).

Now, $P_{1,1}^0 + P_{2,2}^0 + ... + P_{S,S}^0$ counts the number of z with x in $R_0, ..., R_s$.

which equal to the elements of Ω.

Lemma 4.1.1[A.E.Brouwer, A.MCohen and A.Neumaier,1989]

Let Ω be a finite set and let $\Delta = \{R_0, R_1, ..., R_s\}$ be a partition of Ω. Then

(i) $P_{oj}^k = d_{jk}$, $d_{jk} = \begin{cases} 1 & if \quad j = k \\ 0 & if \quad j \neq k \end{cases}$

(ii) $P_{ij}^{0} = d_{ij} n_j$

(iii) $P_{ij}^{k} = P_{ji}^{k}$

(iv) $P_{ij}^{k} n_k = P_{ik}^{j} n_j$

(v) $\sum_j P_{ij}^{k} = n_i$

(vi) $\sum_{\mathbf{l}} P_{ij}^{\mathbf{l}} P_{\mathbf{l}k}^{m} = \sum_r P_{ir}^{m} P_{jk}^{r}$

Proof :

(i) Let $(x,y) \in R_k$,we want to count the number of $z \in \Omega$ such that $(x,x) \in R_0$ and $(y,z) \in R_j$. Now if $Z = X$ (i.e.$(y,x) \in R_j$) this implies that the number of $z \in \Omega$ is one and so $R_j = R_k$, in other words , $j = k$. If $Z \neq X$ $\left(i.e.(y,x) \notin R_j\right)$. Then the number of $z \in \Omega$ is zero and this implies $j \neq k$. Thus $P_{oj}^{k} = d_{jk}$.

(ii) Let $(x,x) \in R_0$. We want to count the number of $z \in \Omega$ such that $(x,z) \in R_i$ and $(y,z) \in R_j$ if $i \neq j$ and $(x,y) \in R_i \Rightarrow (x,z) \notin R_j$ and so the number of $z \in \Omega$ is equal to zero. If $i = j$, the number of $z \in \Omega$ with $(x,z) \in R_i$ is n_i .

(iii) Let $(x,y) \in R_k$. Now P_{ij}^{k} is the number of $z \in \Omega$ such that $(x,z) \in R_i$ and $(y,z) \in R_j$ which is the same as the number of $z \in \Omega$ such that $(x,z) \in R_j$ and $(x,z) \in R_j$ and $(y,z) \in R_i$ (that is P_{ji}^{k}).

(iv) Rewrite the equation as $P_{kk}^{o} P_{ij}^{k} = P_{jj}^{o} P_{ik}^{j}$. Now let $\left(x, y_s\right) \in R_k$

$s = 1, \ldots, r$. Then $P_{kk}^{o} P_{ij}^{k}$ is equal to the number of $z \in \Omega$ such that for any pair $\left(x, y_1\right) \in R_k$ with $(x,z) \in R_i$ and $\left(y_1, z\right) \in R_j$ + number of $z \in \Omega$, $\left(x, y_2\right) \in R_k$ with $(x,z) \in R_i$ and $\left(y_2, z\right) \in R_j$ + …+ number of $z \in \Omega$, $\left(x, y_r\right) \in R_r$ with $(x,z) \in R_i$ and $\left(y_r, z\right) \in R_j$ and this is the same $n_j P_{ik}^{j}$.

(v) Let $(x,y)\in R_k$ we want to count the number of $z\in\Omega$ with $(x,z)\in R_i$ and $(y,z)\in R_o$ + the number of Z with $(x,z)\in R_i$ and $(y,z)\in R_1$ + … + number of Z with $(x,z)\in R_i$ and $(y,z)\in R_{j-1}$.Thus the summation will be the number of $z\in\Omega$ with $(x,z)\in R_i$ which is independent of $x\in\Omega$ which equal n_i .

(vi) The expression at both sides count quadruples (z , u , x ,y) with $(z,u)\in R_i$, $(u,z)\in R_j$ and $(x,y)\in R_k$ for a fixed pair $(z,y)\in R_m$.

4.2 Representation of association schemes by matrices

The relation R_i of symmetric association scheme are represented by their (0,1)- adjacency matrices A_i as follows

$$(A_i)_{x,y}=\begin{cases}1 \quad , \quad if \quad (x,y) \quad \in R_i \\ 0 \quad , \quad otherwise\end{cases}.$$

The definition of an association scheme is equivalent to saying that the A_i are $n\times n$ $(0,1)$ -matrices; where $|V(\Gamma)|=n$ which satisfy :

(i) A_i is symmetric

(ii) $\sum_{i=0}^{n}A_i=J$, the all –ones matrix

(iii) $A_0=\mathrm{I}$ where I is the multiplicative identity matrix

(iv) $A_iA_j=\sum_{k=0}^{n}P_{ij}^{k}A_k=A_jA_i \quad ,i,j=0,\dots,n$

Example 4.2.1: A strongly regular graph is 2-association scheme where A_1 and A_2 denote the adjacency matrices of the graph and its complement

In the next example we use the distance $d(u,v)$ of the vertices u and v of a graph, and the relations R_i defined by $(u,v)\in R_i$ if and only if $d(u,v)=i$ for $i=0,\dots,s$

Example 4.2.2 : The Hexagon

Gives rise to a 3-association scheme , since the distance i matrices A_i

$$A_1=\begin{bmatrix}0 & J-\mathrm{I}\\ J-\mathrm{I} & 0\end{bmatrix},$$

$$A_2=\begin{bmatrix}J-\mathrm{I} & 0\\ 0 & J-\mathrm{I}\end{bmatrix}$$

$$A_0 = \begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix}$$

Theorem 4.2.1: Let Δ be a partition of $\Omega \times \Omega$ into s classes, let A_i, A_j be the adjacency matrix representing R_i, R_j then $A_i A_j = A_j A_i$.

Proof: Let A_i represent R_i and A_j represent R_j

Since, $P_{ij}^k = P_{ji}^k$ and $A_i = A_i^T$

$$(A_i A_j)^T = A_j^T A_i^T = \sum_{k=0}^{s} P_{ij}^k A_k$$

$$= \sum_{k=0}^{s} P_{ji}^k A_k$$

$$= A_j A_i .$$

4.3 Eigenmatrices of the associated schemes

By the previous section the matrices A_i can be diagonalized simultaneously; a decomposition of c^n was found as a direct sum of $d+1$ eigenspaces at dimension m_i $(0 \le i \le d)$. One of the eigenspaces has dimension 1, since J is an adjacency algebra and J has n as an eigenvalue of multiplicity 1, so suppose $m_o = 1$. The m_i is called the multiplicity of the scheme; if $\{E_j : 0 \le j \le d\}$ is the basis of minimal idempotent of the adjacency algebra. Then

$rkE_j = trE_j$,where rk stands for rank,

$\sum_{j=0}^{d} E_j = I$, and

$E_0 = n^{-1} J$

Definition 4.3.1: Let $p_{i,j}$ be the eigenvalues of A_j occurring with multiplicities m_i. Let $Q_{ij} = \frac{m_j}{n_i} p_{ji}$. The matrices $P = [p_{ij}]$, $Q = [Q_{ij}]$

for i,j = 0,1,…,d are called the eigenmatrices of the scheme.

Example 4.3.1: Let $\Omega = \{1, 2, ..., 10\}$,and let $R = \{A_0, A_1, A_2\}$ where $A_0 = I_{10}$.

$$A_1 = \begin{bmatrix} 0&0&0&0&1&0&1&0&1&0 \\ 0&0&1&0&0&1&1&0&0&0 \\ 0&1&0&1&0&0&0&0&1&0 \\ 0&0&1&0&1&0&0&1&0&0 \\ 1&0&0&1&0&1&0&0&0&0 \\ 0&1&0&0&1&0&0&0&0&1 \\ 1&1&0&0&0&0&0&1&0&0 \\ 0&0&0&1&0&0&1&0&0&1 \\ 1&0&1&0&0&0&0&0&0&1 \\ 0&0&0&0&0&1&0&1&1&0 \end{bmatrix}$$

$$A_2 = \begin{bmatrix} 0&1&1&1&0&1&0&1&0&1 \\ 1&0&0&1&1&0&0&1&1&1 \\ 1&0&0&0&1&1&1&1&0&1 \\ 1&1&0&0&0&1&1&0&1&1 \\ 0&1&1&0&0&0&1&1&1&1 \\ 1&0&1&1&0&0&1&1&1&0 \\ 0&0&1&1&1&1&0&0&1&1 \\ 1&1&1&0&1&1&0&0&1&0 \\ 0&1&0&1&1&1&1&1&0&0 \\ 1&1&1&1&1&0&1&0&0&0 \end{bmatrix}$$

$sp(A_0) = \{1^1, 1^4, 1^5\}$, $sp(A_1) = \{3^1, -2^4, 1^5\}$ and $sp(A_2) = \{6^1, 1^4, -2^5\}$

$n_0 = 1$, $n_1 = 3$, $n_2 = 6$. Then p and Q are

$$p = \begin{bmatrix} 1 & 3 & 6 \\ 1 & -2 & 1 \\ 1 & 1 & -2 \end{bmatrix}, \qquad Q = \begin{bmatrix} 1 & 4 & 5 \\ 1 & \frac{-8}{3} & \frac{5}{3} \\ 1 & \frac{4}{6} & \frac{-10}{6} \end{bmatrix}$$

Lemma 4.3.1[E.Bannai and T.Ito 1984]

Let P and Q be the eigenmatrices of the associated scheme. Then,

(i) $A_j = \sum_{i=0}^{d} p_{ij} E_i$

(ii) $E_j = \frac{1}{n} \sum_{i=0}^{d} Q_{ij} A_i$

(iii) $PQ = QP = n\mathrm{I}$

(iv) $A_j E_i = P_{ij} E_i$

Lemma 4.3.2[A.E.Brouwer, A.M Cohen and A.Neumaier 1989]
The matrices P and Q of association scheme satisfy :

(i) $P_{i0} = 1$, $Q_{i0} = 1$

(ii) $P_{0j} = n_j$, $Q_{0j} = m_j$

(iii) $P_{ij} P_{ik} = \sum_{\ell} P^{\ell}{}_{jk} P_{i\ell}$

(iv) $P_{ij} Q_{ki} = \sum_{\ell} P^{k}{}_{j\ell} Q_{\ell i}$

(v) $\left|P_{ij}\right| \le n_j$, $\left|Q_{ij}\right| \le m_j$

(vi) $\sum_{j} P_{jk} = \sum_{i} P^{i}_{ik}$.

Proof : (i) By Lemmann4.3.1

$AE_i = P_{ij} E_i$,so $A_0 E_i = P_{i0} E_i$, $\mathrm{I}E_i = P_{i0} E_i$. Thus, $P_{i0} = 1$.

By the same Lemma

$E_j = \frac{1}{n}\sum_{i=0}^{d} Q_{ij} A_i$. Hence, $E_0 = \frac{1}{n}\sum_{i=0}^{d} Q_{i0} A_i$. But $E_0 = n^{-1} J$, $\sum_{i=0}^{d} A_i = J$.

Thus, $\frac{J}{n} = \frac{1}{n}\sum_{i=0}^{d} Q_{i0} A_i$ implies $Q_{i0} = 1$.

(ii) $A_j E_i = P_{ij} E_i$, $A_j E_0 = P_{0j} E_0$,

Since, $E_0 = \frac{J}{n}$, we see that $A_j J = P_{0j} J$, but $A_j J = n_j J$, and so $P_{0j} = n_j$.

Now, $E_j = \frac{1}{n}\sum_{i=0}^{d} Q_{ij} A_j$ and trace $E_j = rkE_j = m_i$. Thus $Q_{0j} = m_i$.

(iii) By Theorem 4.2.1

$$A_i A_k = \sum_{\ell} P_{ik}{}^{\ell} A_{\ell}$$

Now, $A_i A_k = P_{ij} P_{ik} E_i E_i^{-1}$, and $A_{\mathbf{l}} = P_{i\mathbf{l}} \mathrm{I}$. Hence, $P_{ij} P_{ik} = \sum_{\mathbf{l}} P_{jk}^{\mathbf{l}} P_{i\mathbf{l}}$.

(iv) By (iii)

$$P_{ij} P_{ik} = \sum_{\mathbf{l}} P_{jk}^{\mathbf{l}} P_{i\mathbf{l}}$$

Now, by the definition of Q and P

$$P_{ij} \frac{n_k Q_{ki}}{m_i} = \sum P_{jk}^{\mathbf{l}} \frac{n_{\mathbf{l}} Q_{\mathbf{l}i}}{m_i}$$

Thus, $n_k P_{ij} Q_{ki} = \sum_{\mathbf{l}} P_{j\mathbf{l}}^{k} n_k Q_{\mathbf{l}i}$

and so, $P_{ij} Q_{ki} = \sum_{\mathbf{l}} P_{j\mathbf{l}}^{k} Q_{\mathbf{l}i}$.

(v) The greatest eigenvalue of A_i is equal to the valencies n_i, the second part follows from part one and the definition of Q and P .

(vi) $n \sum_j P_{ik} = \sum_{i,j} Q_{ij} P_{ji} P_{jk} = \sum_{i,j,\mathbf{l}} P_{ik}^{\mathbf{l}} Q_{ij} P_{j\mathbf{l}}$

But, $\sum_{i,j,\mathbf{l}} P_{ik}^{\mathbf{l}} Q_{ij} P_{j\mathbf{l}} = n \sum_i P_{ik}^{i}$

Thus, $n \sum_j P_{jk} = n \sum_i P_{ik}^{i}$,

and so, $\sum_j P_{jk} = \sum_i P_{ik}^{i}$, .

In the following Lemma we will give some new relations between the parameters n_i, n, m_i, P_{ij}, $P_{ij}{}^{k}$, Q_{ij} and the matrices A_i, E_j, P and Q .

Lemma 4.3.3 : The parameters n_i, m_i, P_{ij}, $P_{ij}{}^{k}$, Q_{ij} and the matrices A_i, E_j, P and Q . Satisfy :

(i) $E_i = \frac{m_i}{n} \sum_{j=0}^{d} \left(P_{ij} \big/ n_j \right) A_j$

(ii) $P_{0j} = \sum_{i=0}^{d} P_{ij}^{k}$

(iii) $P_{ij}{}^{0} = d_{ij} P_{0j}$

(iv) $\frac{P_{ij}^{k}}{P_{ik}^{k}} = \frac{P_{0j}}{P_{0k}}$

(v) $P_{ij}^{k} \le \min(n_i, n_j)$

(vi) $\sum_j P_{0j} = n, \ \sum_j Q_{0j} = n,$

(vii) $\sum_{j=0}^{d} P_{ij} = 0, \ 0 < i < d$

(viii) $\sum_i P_{ik}^{i}$ = summation of distinct eigenvalues of A_k

Proof : (i) By the definition of $Q_{ji} = \frac{m_i}{n_j} P_{ij}$, and by Lemma 4.3.1(ii)

Hence $E_i = \frac{m_i}{n} \sum_{j=0}^{d} \left(P_{ij} / n_j \right) A_j$ $E_i = \frac{1}{n} \sum_{j=0}^{d} Q_{ji} A_j$,so $E_i = \frac{1}{n} \sum_{j=0}^{d} \frac{m_i}{n_j} P_{ij} A_j$.

(ii) By Lemma 4.3.2 (ii) $p_{0j} = n_j$, and by Lemma 4.1.1 (v)

$\sum_{i=0}^{d} P_{ij}^{k} = n_i$. Thus, $n_j = \sum_{i=0}^{d} P_{ji}^{k}$, but $P_{ij}^{k} = P_{ji}^{k}$. Hence, $\sum_{i=0}^{d} P_{ij}^{k} = n_j = P_{0j}$.

(iii) By Lemma 4.1.1 (ii) $P_{ij}^{0} = d_{ij} n_j$,and by Lemma 4.3.2 $P_{0j} = n_j$.

(iv) By Lemma 4.1.1. $P_{ij}^{k} n_k = P_{ik}^{j} n_j$,and by Lemma 4.3.2. $n_j = P_{oj}$, $n_k = P_{ok}$.

(v) P_{ij}^{k} = number of $x \in \Omega,$ with $(x,z) \in R_i$ and $(z,y) \in R_j$ for any pair $(x,y) \in R_k$ .Now the number of $z \in \Omega,$ with $(x,z) \in R_i$ is n_i . Which is independent of $x \in \Omega,$ and the number of $z \in \Omega,$ with $(z,y) \in R_j$ is n_j . Hence, the number of $z \in \Omega,$ with $(x,z) \in R_i$ and $(z,y) \in R_j$ will be less than or equal to $\min(n_i, n_j)$.

(vi) Now columns j in the matrix P $\{j = 0, \ldots, d\}$ represent the eigenvalues of A_j .(i.e $\sum_j P_{0j}$ = summation of the first row in P), which equals summation of eigenvalues of A_j with multiplicity one. Thus,

$$\sum_j P_{0j} = \sum_j n_j = n$$

Similarly $\sum_j Q_{0j}$ = summation of the multiplicities of A_j .

(vii) Let P be the eigenmatrix of the association scheme (Ω, Δ).

Thus $\sum_{j=0}^{d} P_{ij}$,(where i >0), is the summation of the entries of i-th row in the matrix P .

$$P = \begin{bmatrix} P_{00} & P_{01} & \dots & P_{0d} \\ P_{10} & P_{11} & \dots & P_{1d} \\ \dots & \dots & \dots & \dots \\ P_{d1} & P_{d2} & \dots & P_{dd} \end{bmatrix}$$

(i.e. summation of eigenvalues of A_j with $m_i > 1$).

(viii) By Lemma 4.3.2 (vi) $\sum_j P_{jk} = \sum_i P_{ik}^i$. Thus by the definition of the eigenmatrices $\sum_i P_{ik}{}^i$ =summation of distinct eigenvalues of A_k .

4.4 An application

Definition 4.4.1 The i-th intersection matrix, denoted by L_i ,is the matrix whose (k, j)- entry is $P_{ij}{}^k$.

Lemma 4.4.1: Let L_i be an intersection matrix and p the eigenmatrix of the association scheme(Ω,Δ). Then $PL_i P^{-1} = diag\left(p_{0i}, p_{1i}, \dots, p_{di}\right)$

Proof : Let $B = diag\left(p_{0i}, p_{1i}, \dots, p_{di}\right)$. We want to show $PL_i P^{-1} = B$, by showing $PL_i = BP$, $i \in \{0, \dots, d\}$. The matrix PL_i is the matrix whose $(\mathbf{l}, j)$- entry is $\sum_k P_{\mathbf{l}k} P_{ij}^k$,where

$$P = \begin{bmatrix} P_{00} & P_{01} & \dots & P_{0d} \\ P_{10} & P_{11} & \dots & P_{1d} \\ \dots & \dots & \dots & \dots \\ P_{d0} & P_{d1} & \dots & P_{dd} \end{bmatrix} \text{ and } L_i = \begin{bmatrix} P_{i0}{}^0 & P_{i1}{}^0 & \dots & P_{id}{}^0 \\ P_{i0}{}^1 & P_{i1}{}^1 & \dots & P_{id}{}^1 \\ \dots & \dots & \dots & \dots \\ P_{i0}{}^d & P_{i1}{}^d & \dots & P_{id}{}^d \end{bmatrix}$$

and the matrix BP is the matrix whose $(\mathbf{l}, j)$- entry is $p_{\mathbf{l}i} p_{\mathbf{l}j}$.

Since $p_{\mathbf{l}i} p_{\mathbf{l}j} = \sum_k P_{\mathbf{l}k} P_{ij}^k$,by virtue of Lemma 4.3.2 (iii),we have

$PL_i P^{-1} = B$.

Example 4.4.1 : Let (Ω,Δ)as in Example 4.3.1. Let

$$L_1 = \begin{bmatrix} 0 & 3 & 0 \\ 1 & 0 & 2 \\ 0 & 1 & 2 \end{bmatrix}$$

$$B = diag\left(p_{01}, p_{11}, p_{21}\right) = \begin{bmatrix} 3 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \text{ and } L_2 = \begin{bmatrix} 0 & 0 & 6 \\ 0 & 2 & 4 \\ 1 & 2 & 3 \end{bmatrix}$$

Then,

$$B = diag\left(p_{02}, p_{12}, p_{22}\right) = \begin{bmatrix} 6 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{bmatrix}$$

Remark 4.4.1 : By Lemma 4.4.1, the P_{hi} are eigenvalues of L_i, and it follows that $A_i \to L_i$ defines the isomorphism between the algebra of the scheme and the algebra generated by L_i.

Note 4.4.1: If the intersection matrices are known, it is possible to compute the eigenmatrices P and Q and so the eigenvalues of A_i.

Lemma 4.4.2: Let Δ_n , Δ_m be the diagonal matrices, $diag\left(n_0, n_1, ..., n_d\right)$

and $diag\left(m_0, m_1, ..., m_d\right)$ respectively. Then,

$$\Delta_n Q = \left(\Delta_m P\right)^T .$$

Proof : Let $Q = \left[Q_{ij}\right]$, $P = \left[P_{ij}\right]$

Now, the (i,j)- entry of the matrix $\Delta_n Q$ is $n_i Q_{ij}$ and the (i,j)- entry of the matrix $\Delta_m P$ is $m_i P_{ij}$.

the (i,j)- entry of $(\Delta_m P)^T = m_j P_{ji}$. By virtue of the definition of the eigenmatrices P and Q, $n_i Q_{ij} = m_j P_{ji}$.

Hence, $\Delta_n Q = (\Delta_m P)^T$.

Lemma 4.4.3: Let L_i be an intersection matrix and let $\Delta_n = diag\left(n_0, n_1, ..., n_d\right)$. Then, $\Delta_n L_i$ is symmetric.

Proof : The (k,j)- entry of $\Delta_n L_i = n_k P_{ij}^k$, and the (k,j)- entry of $(\Delta_n L_i)^T = n_i P_{ik}^j$. Since $n_k P_{ij}^k = n_i P_{ik}^j$ by Lemma 4.1.1(iv).

Thus , $\Delta_n L_i = (\Delta_n L_i)^T$.

Lemma 4.4.4: Let u be a right eigenvector of L_i corresponding to an eigenvalue l . Then $(\Delta_n u)^T$ is a left eigenvector of L_i .

Proof : We want to show that $(\Delta_n u)^T L_i = (\Delta_n u)^T l$.

Since, u is a right eigenvector of L_i , we have

$L_i u = lu$, $\Delta_n L_i u = \Delta_n lu$ (multiplying both sides by Δ_n).

$(\Delta_n L_i u)^T = (\Delta_n lu)^T$ (taking transpose of both sides)

$u^T L_i^T \Delta_n^T = u^T l \Delta_n^T$

$u^T (\Delta_n L_i)^T = u^T l \Delta_n^T$

(since $\Delta_n L_i = (\Delta_n L_i)^T$), then $u^T \Delta_n L_i = (\Delta_n u)^T l$ and

$u^T \Delta_n^T L_i = (\Delta_n u)^T l$ since $\Delta_n = \Delta_n^T$, and so $(\Delta_n u)^T L_i = (\Delta_n u)^T l$.

Lemma 4.4.5: Let e_h be the h^{th} unit vector . Then

, $i = 0, \ldots, d$ $L_i Q e_h = P_{hi} Q e_h$

(i.e. the common right eigenvectors of the L_i are Qe_h).

Proof : Let $e_h = \begin{bmatrix} 0 & 0 & .. & 1 & 0 & .. & 0 \end{bmatrix}$;where the 1 is position h.

$$Qe_h = \begin{bmatrix} Q_{0h} \\ \vdots \\ Q_{dh} \end{bmatrix}$$

Since, L_i is the matrix whose (k,j)-entry is $P_{ij}^{\ k}$, we have $L_i Qe_h$ is a matrix whose (k,h)- entry is $\sum_{l} P_{il}^{\ k} Q_{lh}$. By virtue of Lemma 4.3.2 (iv) $\sum_{l} P_{il}^{\ k} Q_{lh} = P_{hi} Q_{kh}$.

Since $P_{hi} Q_{kh}$ is the entry (k,h) of the matrix $P_{hi} Qe_h$. Thus $L_i Qe_h = P_{hi} Qe_h$.

Remark 4.4.2: $(Qe_h)_0 = (\Delta_n Qe_h)_0$.

Proof : The entry in 0-position in Qe_h is Q_{0h} . Because $m_i P_{ij} = n_j Q_{ji}$, we

Have $Q_{0h} = \frac{m_h P_{h0}}{n_0}$. Now, P_{h0} is the eigenvalue of A_0 with multiplicity h and this implies $P_{h0} = 1$, and $n_0 = 1$ and so $P_{h0} = m_h$. On the other hand, $(\Delta_n Qe_h)_0 = m_h$. Since, $\Delta_n = diag(n_0, n_1, ..., n_d)$, $n_0 = 1$ and the entry in the 0-position of $\Delta_n Qe_h$ is Q_{0h} .

Remark 4.4.3: The inner product $(\Delta_n Qe_h, Qe_i) = d_{ih} m_h n$.

Proof: The inner product

$$(\Delta_n Qe_h, Qe_i) = \sum_s n_s Q_{sh} Q_{si}$$

$$\sum_s n_s Q_{sh} Q_{si} = nm_h d_{ih}$$

(by virtue of Lemma 4.3.2, and the definition of the eigenmatrices).

Thus $(\Delta_n Qe_h, Qe_i) = nm_h d_{ih}$.

We now state Proposition 4.4.1 from Brouwer, Cohen and Neumaier book

This proposition is false and we give a counter-example .

Proposition 4.4.1 [A.E.Brouwer, A.M Cohen and A.Neumaier,1989]

Let u_i and $v_i^T = (\Delta_n u_i)^T$ (i=0 , … , d) be the common right and left eigenvectors of the matrices L_i, respectively, normalized such that $(u_i)_0 = (v_i)_0 = 1$. Then $m_i = \frac{n}{(u_i, v_i)}$ (i=0 , … , d) the rows of P are the v_i^T and the columns of Q are the $m_i u_i$. $i = 0, \mathbf{K}, d$.

Counter – Example 4.4.2:

Let (Ω, Δ) be an association scheme as in Example 4.3.1.

The eigenmatrices P and Q are

$$P = \begin{bmatrix} 1 & 3 & 6 \\ 1 & -2 & 1 \\ 1 & 1 & -2 \end{bmatrix}$$

$$Q = \begin{bmatrix} 1 & 4 & 5 \\ 1 & \frac{-8}{3} & \frac{5}{3} \\ 1 & \frac{4}{6} & \frac{-10}{6} \end{bmatrix}$$

The intersection matrices L_i are

$$L_0 = \mathrm{I}_3\ ,\ L_1 = \begin{bmatrix} 0 & 3 & 0 \\ 1 & 0 & 2 \\ 0 & 1 & 2 \end{bmatrix} \quad , L_2 = \begin{bmatrix} 0 & 0 & 6 \\ 0 & 2 & 4 \\ 1 & 2 & 3 \end{bmatrix}$$

$PL_1P^{-1} = diag\,(3,-2,1)$ and $PL_2P^{-1} = diag\,(6,1,-2)$.

$\Delta_n = diag\,(1,3,6),\ \Delta_m = diag\,(1,4,5)$

Now, by Lemma 4.4.5 Qe_i is the common right eigenvector and $(\Delta_n Qe_i)^T$ is the common left eigenvector of L_j (i.e. $u_i = Qe_i$ and $v_i^T = (\Delta_n Qe_i)^T$.

Now, $e_1 = [1 \quad 0 \quad 0]^T$

$e_2 = [0 \quad 1 \quad 0]^T$, $e_3 = [0 \quad 0 \quad 1]^T$.

Thus,

$u_1 = [1 \quad 1 \quad 1]^T$; $m_1 u_1 = [1 \quad 1 \quad 1]^T$;

$$u_2 = \begin{bmatrix} 4 & \frac{-8}{3} & \frac{4}{6} \end{bmatrix}^T ;\ m_2 u_2 = \begin{bmatrix} 16 & \frac{-32}{3} & \frac{16}{6} \end{bmatrix}^T ;$$

$$u_3 = \begin{bmatrix} 5 & \frac{5}{3} & \frac{-10}{6} \end{bmatrix}^T ;\ m_3 u_3 = \begin{bmatrix} 25 & \frac{25}{3} & \frac{-50}{6} \end{bmatrix}^T ;$$

$v_1^T = [1 \quad 3 \quad 6]$,

$v_2^T = [4 \quad -8 \quad 4]$,

$v_3^T = [5 \quad 5 \quad -10]$

Hence, v_i^T are not the rows of the eigenmatrix P , and also $m_i u_i$ are not the columns of the matrix Q and $m_i \neq \frac{n}{(u_i, v_i)}$.

Now, we will give the correct form of Proposition 4.4.1 which plays an important role in the applications of the association scheme in the spectra of graphs.

Proposition 4.4.2: Let u_i and $v_i^T = (\Delta_n u_i)^T$ $i = 0, \ldots, d$ be the common right and left eigenvectors of the intersection matrices L_j, respectively, normalized such that $(u_i)_0 = (v_i)_0 = 1$. Then $m_i = \frac{(u_i, v_i)}{n}$, where $n = |\Omega| = \sum_{i=0}^{d} n_i$, $i = 0, \ldots, d$ the rows of P are the $\frac{v_i^T}{m_i}$ $i = 0, \ldots, d$, and the columns of Q are u_i $i = 0, \ldots, d$.

Proof: By Lemmas 4.4.4 and 4.4.5, $(\Delta_n Q e_i)^T$ ($i = 0, \ldots, d$) are common left eigenvectors of L_j . Hence, $v_i^T = \left(\Delta_n Q e_i\right)^T$, but

$(\Delta_n Q e_i)^T = e_i^T \Delta_m P$ by Lemma 4.4.2

$$= [m_i P_{i0} \quad m_i P_{i1} \quad \ldots \quad m_i P_{id}]$$

$$= m_i \, [P_{i0} \quad P_{i1} \quad \ldots \quad P_{id}]$$

Thus, $\frac{1}{m_i}(\Delta_n Q e_i)^T = [P_{i0} \quad P_{i1} \quad \ldots \quad P_{id}]$

and so, $\frac{v_i^T}{m_i}$ ($i = 0, \ldots, d$) are the rows of the eigenmatrix P .

Now, $Q = \left[Q_{ij}\right]$ and Qe_i are common right eigenvectors of L_j by Lemma4.4.5 .

Since, $Qe_i = [Q_{0i} \quad Q_{1i} \quad \ldots \quad Q_{di}]^T$,we have $Qe_i = u_i$ ($i = 0, \ldots, d$) are the columns of the eigenmatrix Q . Now, $Qe_i = u_i$ and $\Delta_n Q e_i = v_i$, and by Remark 4.4.3 $(u_i, v_i) = d_{ii} m_i n$. Thus $(u_i, v_i) = n m_i$,(i.e. $m_i = \frac{(u_i, v_i)}{n}$) .

Example 4.4.3: Let (Ω, Δ) be an association scheme as in Example 4.3.1

$$L_1 = \begin{bmatrix} 0 & 3 & 0 \\ 1 & 0 & 2 \\ 0 & 1 & 2 \end{bmatrix}$$

$$u_1 = [1 \quad 1 \quad 1]^T$$

$$u_2 = \begin{bmatrix} 4 & \frac{-8}{3} & \frac{4}{6} \end{bmatrix}^T$$

$$v_1^T = [1 \quad 3 \quad 6], \ \frac{v_1^T}{m_1} = [1 \quad 3 \quad 6]$$

$$v_2^T = [4 \quad -8 \quad 4], \ \frac{v_2^T}{m_2} = [1 \quad -2 \quad 1],$$

$$v_3^T = [5 \quad 5 \quad -10], \frac{v_3^T}{m_3} = [1 \quad 1 \quad -2],$$

$$m_i = \frac{(u_i, v_i)}{n}$$

$$m_1 = \frac{10}{10} = 1$$

$$m_2 = \frac{40}{10} = 4$$

$$m_3 = \frac{50}{10} = 5$$

Thus,

$$P = \begin{bmatrix} 1 & 3 & 6 \\ 1 & -2 & 1 \\ 1 & 1 & -2 \end{bmatrix} \text{ and } Q = \begin{bmatrix} 1 & 4 & 5 \\ 1 & \frac{-8}{3} & \frac{5}{3} \\ 1 & \frac{4}{6} & \frac{-10}{6} \end{bmatrix}.$$

4.5 Conclusion

The matrix theory in general and graph spectra can be considered as a method of using algebra as a tool in graph theory and its applications; this makes the handling of graph theory easier. In general graph spectra and associated schemes cannot be converted purely into a simple form of matrix theory. We investigate the basic properties of graph spectrum and investigate (symmetric) association schemes, such schemes are strongly related with graphical properties of graphs tangible results and illustrated examples concerning the relation between graph spectra and graphical structure is give.

We have constructed the relations between and among intersection numbers of association schemes and eigenmatrices ,also we introduced a construction of association schemes .

We discussed the graph spectrum and the relations between graphical properties and the related characteristic polynomial of graphs .

Suggested problem

Study the, relation between and among intersection numbers of association schemes and eigenmatrices and it introduces a construction of association schemes through permutation groups. The set of all automorphisms of a graph Γ, with multiplication is the automorphism group $Aut(\Gamma)$.

An automorphism of a graph Γ with n vertices and m edges is (G, H) of permutation matrices such that $G.S = S.H$ where S is the incidence matrix of Γ, G is an $n \times n$ matrix and H is an $m \times m$ matrix, these matrices represent permutations of the vertex and edge sets which preserve incidence.

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